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A SHORT PROOF OF DE SHALIT'S CUP PRODUCT FORMULA

BY

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ABSTRACT

We give a short proof of a formula of de Shalit, expressing the cup product of two vector-valued one-forms of the second kind on a Mumford curve in terms of Coleman integrals and residues. The proof uses the notion of double indices on curves and their reciprocity laws.

1. Introduction

In [dS88] de Shalit proved a formula for the cup product of two vector-valued differential forms on a Mumford curve. This is based on an earlier partial result of his [dS89] for two holomorphic differentials. This formula was later reproved by Iovita and Spiess [IS03]. The goal of this short note is to give an alternative short proof of de Shalit's formula based on the theory of the double index [Bes00, Section 4].

Let us state de Shalit's result. Let K be a finite extension of $\mathbb{Q}_p$. Consider a Mumford curve $\mathcal{H}/\Gamma$, where $\Gamma \subset \mathrm{PGL}_2(K)$ is a Schottky group and $\mathcal{H} \subset \mathbb{P}^1_K$ is the rigid analytic space obtained by removing the limit points of Γ. Let V be a finite-dimensional K-vector space with a representation of Γ. The group Γ acts on the space of V-valued differential forms on $\mathcal{H}$, $\Omega^1(\mathcal{H}, V)$, by the rule

$$\gamma(\sum \omega_i v_i) = \sum (\gamma^{-1})^* \omega_i \gamma(v_i)$$

(compare [dS88,1.1]). We let it act by the same formula on spaces of functions. A V-valued differential one-form ω on $\mathcal{H}$ with values in V is Γ-invariant if

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$\gamma(\omega) = \omega$ for every $\gamma \in \Gamma$. It is of the second kind if its residues (with values in V, computed coordinatewise, in any basis) are 0 at any point $z \in \mathcal{H}$. Let $\langle\ \rangle$ be a Γ-invariant bilinear form on V. The cup product of two Γ-invariant V-valued one-forms of the second kind ω and η can be described by the formula

$$\omega \cup \eta = \sum_{z \in \Gamma \backslash \mathcal{H}} \operatorname{Res}_z \langle F_\omega, \eta \rangle,$$

where F_ω is any primitive of ω locally near z, which exists (formally) because the residue of ω at z is 0, and is independent of the choice of the primitive because the residue of η at z is 0. Note that the expression to be summed indeed depends only on z modulo Γ.

An open annulus is a rigid space isomorphic to the space $s < |z| < r$. An orientation on an annulus may be described as a choice of a parameter z as above, with two parameters considered equivalent if they give the same residue, as defined below. An annulus together with an orientation is called an oriented annulus. A differential form ω on an oriented annulus e has a residue $\operatorname{Res}_e \omega$ such that $\operatorname{Res} \sum a_i z^i dz = a_{-1}$. It can be shown that there are only two orientations, giving residues differing by multiplication by -1. By choosing a basis for V the residue extends easily to V-valued differential forms.

We now recall [dS89, Definition 2.5] that the action of Γ on $\mathcal{H}$ has a good fundamental domain in the following sense: There are pairwise disjoint closed K-rational discs B_i and C_i, $i = 1, \dots, g$, open annuli b_i, c_i, and elements $\gamma_i \in \Gamma$, such that the following holds:

(1) The γ_i freely generate Γ.
(2) The unions $B_i \cup b_i$ and $C_i \cup c_i$ are open discs, still pairwise disjoint.
(3) For each i, γ_i maps B_i isomorphically onto the complement of $C_i \cup c_i$ and b_i isomorphically onto c_i.
(4) The complement of $\bigcup_i (B_i \cup b_i \cup C_i)$ is a fundamental domain for Γ.

We give the annuli c_i and b_i the orientation given by the discs C_i and B_i respectively, i.e., one given by parameters extending to $C_i \cup c_i$ and taking the value 0 in C_i (respectively with b_i and B_i). Thus, c_i is oriented in the same way as in [dS88, 1.5] while b_i is oriented in the reversed direction to loc. cit. (the b_i's do not show up in the formula). With this choice, γ_i: $b_i \to c_i$ is orientation reversing.

The formula of de Shalit involves Coleman integration of holomorphic V-valued one-forms. While this can be described in a completely elementary way since we are dealing with subdomains of the projective line [GvdP80, p. 41], we will use the more involved theory of Coleman [CdS88] and adapt it to our case by choosing a basis of V and then integrate coordinate by coordinate. This is clearly independent of the choice of a basis because Coleman integration is linear (up to constant). The key property of Coleman integration is its functoriality. It immediately implies that from the property $\gamma\omega = \omega$ we may deduce that for any $\gamma \in \Gamma$ the function $\gamma(F_\omega) - F_\omega$ is constant. We can now state the main theorem.

THEOREM 1.1 ([dS88, Theorem 1.6]): *With the data above we have*

$$\omega \cup \eta = \sum_i \langle \gamma_i F_\omega - F_\omega, \operatorname{Res}_{c_i} \eta \rangle - \langle \operatorname{Res}_{c_i} \omega, \gamma_i F_\eta - F_\eta \rangle.$$

The main ingredient in the present proof is the theory of double indices and their reciprocity laws on curves [Bes00, Section 4]. We need a very easy extension of this theory to vector-valued differential forms. Once this has been described, the proof is an easy computation.

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2. Double indices of vector-valued differential forms

In this section we describe a rather straightforward generalization of the theory of double indices [Bes00, Section 4] to the case of vector-valued one-forms. The extension is fairly trivial since we consider only constant coefficients. We work over $\mathbb{C}_p$ for convenience.

Let A be either the field of meromorphic functions in the variable z over $\mathbb{C}_p$ or the ring of rigid analytic functions on an annulus $\{r < |z| < s\}$ over $\mathbb{C}_p$. Let $A_{\log} := A[\log(z)]$ and let $A_{\log,1} \subset A_{\log}$ be the subspace of $F \in A_{\log}$ which are linear in $\log(z)$, a condition which is equivalent to $dF \in Adz$.

Definition 2.1 [Bes00, Proposition 4.5]: The double index,

$$\operatorname{ind}(\ ) \colon A_{\log,1} \times A_{\log,1} \to \mathbb{C}_p,$$

is the unique antisymmetric bilinear pairing such that $\operatorname{ind}(F, G) = \operatorname{Res} F dG$, whenever $F \in A$.

Suppose now that C is a proper smooth curve over $\mathbb{C}_p$ with good reduction $X/\overline{F}_p$, and let $\{x_1, \ldots, x_k\}$ be a finite non-empty set of closed points of X. We then consider, following [CdS88], the rigid analytic space U obtained from C by removing discs D_i of the form $|z_i| \leq r$, with $r < 1$, where the reduction of z_i is a local parameter near the point x_i. Let us call these domains **simple domains.** To the disc D_i corresponds the annulus e_i given by the equation $r < |z_i| < 1$, which is contained in U and oriented by z_i.

Choose a branch of the p-adic logarithm. Given a rigid one-form $\omega \in \Omega^1(U)$, Coleman's theory provides us with a unique up to constant, locally analytic function F_ω on U with the property that $dF_\omega = \omega$. Restricted to the annuli e_i these clearly belong to $A_{\log,1}$ and one can therefore define, for two such functions F_ω and F_η, the double index $\mathrm{ind}_{e_i}(F_\omega, F_\eta)$. It follows from [Bes00, Lemma 4.6] that this index depends only on the orientation. One of the main technical results of [Bes00] is the following.

PROPOSITION 2.2 ([Bes00, Proposition 4.10]): *We have*

$$\sum_i \mathrm{ind}_{e_i}(F_\omega, F_\eta) = \Psi(\omega) \cup \Psi(\eta),$$

where $\Psi\colon H^1_{\mathrm{dR}}(U) \to H^1_{\mathrm{dR}}(C)$ *is a certain projection.*

We will only need the following immediate Corollary, which follows because $H^1_{\mathrm{dR}}(\mathbb{P}^1/\mathbb{C}_p) = 0$.

COROLLARY 2.3: *Suppose that* $C = \mathbb{P}^1$. *Then, in the situation above,*

$$\sum_i \mathrm{ind}_{e_i}(F_\omega, F_\eta) = 0.$$

We can now extend the theory to vector-valued differential forms in a rather trivial way. Suppose we are given a finite-dimensional $\mathbb{C}_p$-vector space with a bilinear form $\langle , \rangle$.

Definition 2.4: Choose bases $\{v_i\}$ and $\{u_i\}$ for V. Suppose that the V-valued Coleman functions F_ω and F_η are written as

$$F_\omega = \sum F_{\omega_i} v_i, \quad F_\eta = \sum F_{\eta_j} u_j.$$

Then, the local index $\mathrm{ind}_e(F_\omega, F_\eta)$ is given by

$$\mathrm{ind}_e(F_\omega, F_\eta) = \sum_{i,j} \mathrm{ind}_e(F_{\omega_i}, F_{\eta_j}) \langle v_i, u_j \rangle.$$

It is easy to check, using the bilinearity of ind_e, that this definition does not depend on the choice of bases. An easy consequence of the definitions is the following.

PROPOSITION 2.5: *Suppose that* $\mathrm{Res}_e\,\omega = 0$. *Then* $\mathrm{ind}_e(F_\omega, F_\eta) = \mathrm{Res}_e\langle F_\omega, \eta\rangle$ *while* $\mathrm{ind}_e(F_\eta, F_\omega) = -\mathrm{Res}_e\langle \eta, F_\omega\rangle$.

We now restrict to the case $C = \mathbb{P}^1$ but consider more general subdomains U, obtained by removing closed discs $D_i = \{|z-\alpha_i| \le r_i\}$, including the case of removing a point when $r_i = 0$. For each i we consider an annulus e_i in U surrounding D_i, in such a way that the open discs $D_i \cup e_i$ are still disjoint. We will call the e_i the **annuli ends** of U. It is easy to see that U can be obtained by gluing simple domains $U' \subset \mathbb{P}^1$ along annuli. Note that the U''s are glued along annuli with reversed orientations. Given $\omega \in \Omega^1(U,V)$, one can define its Coleman integral F_ω first on each of the U''s as before and then by adjusting constants along the annuli. The intersection graph of the U''s is a tree, so there is always a way of choosing an integral globally. This construction coincides with the definition of Coleman integrals in [GvdP80].

PROPOSITION 2.6: *In the situation described above we have, for any rigid* V*-valued one-form on* U, $\sum_i \mathrm{ind}_{e_i}(F_\omega, F_\eta) = 0$.

Proof: The case V trivial and U simple is Corollary 2.3. We next consider the case $U = U_1' \cup U_2'$ with U_1' and U_2' glued along an annulus e. Since e has reversed orientations when considered in U_1' and U_2', the double index ind_e has a reverse sign in these two cases by [Bes00, Lemma 4.6]. Thus, the result for U follows from those for U_1' and U_2'. Now, the case of a general U, still with trivial V, follows immediately. The general case now follows immediately by choosing bases. ∎

PROPOSITION 2.7: *Let e be an annulus in $\mathcal{H}$ and let $\gamma \in \Gamma$. For $\omega \in \Omega^1(e,V)$, let F_ω be its integral. Then γF_ω is a Coleman integral of $\gamma(\omega)$ on $\gamma(e)$. Furthermore, if η is another such form, then we have*

$$\mathrm{ind}_e(F_\omega, F_\eta) = \pm\,\mathrm{ind}_{\gamma(e)}(\gamma(F_\omega), \gamma(F_\eta)),$$

depending on whether γ is orientation reversing or preserving.

Proof: We choose a basis $\{v_i\}$ and $\{u_j\}$ of V. Suppose $F_\omega = \sum f_i v_i$ and $F_\eta = \sum g_j u_j$. Then

$$\mathrm{ind}_e(F_\omega, F_\eta) = \sum_{ij} \mathrm{ind}_e(f_i, g_j)\langle v_i, u_j\rangle$$

and

$$\mathrm{ind}_{\gamma(e)}(\gamma(F_\omega), \gamma(F_\eta)) = \sum_{ij} \mathrm{ind}_{\gamma(e)}((\gamma^{-1})^* f_i, (\gamma^{-1})^* g_j)\langle \gamma(v_i), \gamma(u_j)\rangle.$$

But by [Bes00, Lemma 4.6] we have, for each i,

$$\mathrm{ind}_e(f_i, g_j) = \pm \mathrm{ind}_{\gamma(e)}((\gamma^{-1})^* f_i, (\gamma^{-1})^* g_j),$$

depending on whether γ^{-1} is orientation reversing or preserving. Since also $\langle \gamma(v_i), \gamma(u_j) \rangle = \langle v_i, u_j \rangle$, the result follows immediately. ∎

3. The proof

Proof of Theorem 1.1: By the remark after Equation (5) in [dS88] we may assume that b_i and c_i contain no poles of ω and η. Consider the domain $\mathcal{F} = \mathbb{P}^1 - \bigcup_i (B_i \cup C_i)$, which is of the type considered in Section 2, and its annuli ends are the b_i and c_i. It follows from the description of the fundamental domain for Γ that $\mathcal{F} - \bigcup_i (c_i \cup b_i)$ contains exactly one out of every Γ class of every singularity of either forms. Thus,

$$\begin{aligned} \omega \cup \eta = \sum_{x \in \mathcal{F}} \mathrm{Res}_x \langle F_\omega, \eta \rangle &= \sum_{x \in \mathcal{F}} \mathrm{ind}_x (F_\omega, F_\eta) \\ &= -\sum_i (\mathrm{ind}_{b_i}(F_\omega, F_\eta) + \mathrm{ind}_{c_i}(F_\omega, F_\eta)) \end{aligned}$$

where the last equality follows from Proposition 2.6. We now observe that since γ_i is orientation reversing, we have by Proposition 2.7 that $\mathrm{ind}_{b_i}(F_\omega, F_\eta) = -\mathrm{ind}_{c_i}(\gamma_i F_\omega, \gamma_i F_\eta)$. Therefore

$$\begin{aligned} &-(\mathrm{ind}_{b_i}(F_\omega, F_\eta) + \mathrm{ind}_{c_i}(F_\omega, F_\eta)) \\ &\quad = \mathrm{ind}_{c_i}(\gamma_i F_\omega, \gamma_i F_\eta) - \mathrm{ind}_{c_i}(F_\omega, F_\eta) \\ &\quad = \mathrm{ind}_{c_i}(\gamma_i F_\omega - F_\omega, \gamma_i F_\eta) + \mathrm{ind}_{c_i}(F_\omega, \gamma_i F_\eta - F_\eta) \\ &\quad = \mathrm{Res}_{c_i} \langle \gamma_i F_\omega - F_\omega, \gamma_i \eta \rangle - \mathrm{Res}_{c_i} \langle \omega, \gamma_i F_\eta - F_\eta \rangle \quad \text{by Proposition 2.5} \\ &\quad = \langle \gamma_i F_\omega - F_\omega, \mathrm{Res}_{c_i} \eta \rangle - \langle \mathrm{Res}_{c_i} \omega, \gamma_i F_\eta - F_\eta \rangle. \end{aligned}$$

The theorem follows immediately. ∎

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